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Nonequilibrium Dynamics of Quantum Fields in Inflationary Cosmology

S. A. Ramsey[1]

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We summarize a recent study (with B. L. Hu) of the nonequilibrium dynamics of an unbroken-symmetry inflaton field during postinflationary reheating, during which the energy density contained in the expectation value of the inflaton field is rapidly transferred to inhomogeneous quantum modes of the inflaton field. The coupled dynamics of the expectation value (mean field) of a scalar inflaton field with an unbroken global O(N) symmetry and its quantum variance is studied using the leading-order, large-N approximation in a spatially flat Friedmann–Robertson–Walker (FRW) background spacetime. The initial conditions for the mean field, variance, and Hubble parameter were chosen to be consistent with conditions at the end of slow roll in chaotic inflation. Backreaction of the dynamics of the mean field on the spacetime is incorporated self-consistently using the semiclassical Einstein equation. The coupled dynamical equations for the mean field, variance, and scale factor are solved for various choices of the mean field amplitude at the end of the slow-roll period, in order to determine the effect of spacetime curvature on "preheating," the parametric resonance-induced, rapid transfer of energy from the mean field to the inhomogeneous inflaton modes. It is shown that cosmic expansion can dramatically effect the efficiency of preheating in the particular model studied.

1. INTRODUCTION

The central unifying component of inflationary universe scenarios [1–6] is that there is a quantum field, the inflaton field, which contributes a portion ρ of the total energy density of the universe for which the equation of state (for some period of time) is very nearly vacuum-dominated, i.e., the pressure p satisfies $p \simeq -\rho$. If the inflaton's vacuum energy density dominates over other forms of energy at some time in (and in some causally coherent patch of) the early universe, and provided that spatial inhomogeneities are not too

[1] Department of Physics, University of Maryland, College Park, Maryland 20742-4111; e-mail: sramsey@physics. umd.edu.

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large [7–9], a period of exponential growth of the cosmological scale factor $a(t)$ in cosmological (or "comoving") time will ensue.

The exponential growth of the scale factor during a vacuum-dominated era has important consequences for the thermal history of the universe, because for quantum fields which are in local thermal equilibrium in the expanding-universe spacetime, the temperature of such fields decreases in proportion to $1/a(t)$. Furthermore, any small spatial inhomogeneities present in the energy-momentum tensor prior to the onset of inflation are redshifted away during inflation, since spatial gradient energy in a quantum field decreases like $1/a(t)^2$. During the "slow-roll" period of inflation, the scale factor typically must increase by at least a factor of roughly 10^{30} in order to solve the large-scale homogeneity and "unwanted relics" problems [4].

Therefore, at the end of the slow-roll period, to a very good approximation, the quantum state of the inflaton field consists of a coherent state which has a large, spatially homogeneous expectation value $\hat{\phi}$, with fluctuations about $\hat{\phi}$ given by an adiabatic vacuum state [1, 4, 5]. The energy density of the universe is at this point dominated by the energy density of the expectation value $\hat{\phi}$, with all other fields also being very nearly in an adiabatic vacuum state. The subsequent dynamics of the mean field $\hat{\phi}$, in which energy is typically transferred, due to particle production, from $\hat{\phi}$ to both light fields coupled to the inflaton and to the inflaton's own inhomogeneous spatial Fourier modes, is called the *reheating period*. This period includes the reestablishment of local thermal equilibrium via collisional processes, through which the postinflationary Universe evolves into the standard radiation-dominated, big-bang cosmology.

In light of the fact that the mean field $\hat{\phi}$ dominates the energy density of the universe at the end of the slow-roll period, the *reheating problem* can be stated as: At what maximal temperature T_{RH} (presumably commensurate with the redshifted energy density ρ and the number of effective degrees of freedom) and on what time scale τ_{RH} does the universe reestablish local thermal equilibrium? Answering this question is important for several reasons. First, virtually all of the entropy density in the observable universe was produced during the reheating period [4]. Second, there are several areas within cosmology which may place bounds on the reheat temperature, thereby providing constraints for specific inflationary models. In this sense, the reheat temperature is an important parameter characterizing a specific inflationary cosmology.

First, the maximal temperature achieved during the reheating period is important because it is related to the minimum number of e-folds of inflation necessary to solve the large-scale homogeneity and "flatness" problems [4]. The exact nature of the relationship between T_{RH} and the required number N_{min} of e-folds of inflation is model-dependent, but in general such a relation will hold. This is because the scale factor is increasing (albeit as a power of the time t rather than as an exponential) throughout the reheating period.

The reheating temperature is important to baryogenesis. This is because any baryon asymmetry which may have existed prior to inflation will likely be diluted away by the large increase in the scale factor during inflation. Therefore, any baryon asymmetry must be produced either during or after inflation. A reheating temperature as high as T_{GUT} would make possible GUT baryogenesis via out-of-equilibrium decay of a supermassive GUT boson [10]. Furthermore, nonperturbative effects during reheating (during the "preheating" stage of the reheating period) may also make possible the generation of a baryon asymmetry from a nonthermal distribution of GUT bosons produced during preheating [12–15]. If instead the baryon asymmetry was produced during nonequilibrium processes during the electroweak phase transition, the reheating temperature must have been higher than the critical temperature for the electroweak transition [4].

An understanding of the dynamics of the inflaton during the reheating period is also potentially important to the production of topological defects. This is because nonperturbative effects may make possible the nonthermal restoration of a GUT-scale symmetry during the reheating period, which could lead to the production of topological defects during the subsequent cooling and phase transition. This has led some to conclude that inflation does not necessarily "solve" the "unwanted relics" problem [16, 17] (for alternative points of view on symmetry restoration, refs. 18 and 19].

Even if nonthermal symmetry restoration during preheating is not a viable mechanism for generating topological defects [18], an understanding of the reheating period may still be important to the dark matter problem. This is because there is both numerical [11, 20–23, 19, 24–28] and analytical [11, 29, 19, 26, 28] evidence that, in the absence of a symmetry-breaking potential for the inflaton field, and in the absence of Yukawa-type couplings of the inflaton to fermion fields, the inflaton eventually "freezes out" (damping of the amplitude of inflaton mean-field oscillations due to backreaction from particle production ceases, and any subsequent damping is due to gravitational redshift only), leaving a spatially homogeneous condensate of matter-dominated (nonrelativistic) energy density in the inflaton mean field, which has been conjectured as a possible component of cold dark matter [11].

All of the above-mentioned bridges between the physics of the reheating period and potential observational constraints on specific inflationary models depend crucially on the *nonequilibrium dynamics* of quantum fields during the reheating period. The inflaton field during inflation is said to be in disequilibrium because the initial conditions for the inflaton field at the onset of reheating, with virtually all the energy density concentrated in the zero mode (which has a large expectation value), constitute a nonthermal spectrum.

Furthermore, the oscillations of $\hat{\phi}$ are more rapid than the particle production mechanisms which would lead to equilibration of the quantum field. For cases where the inflaton field has a self-interaction (e.g., chaotic inflation with a quartic potential), the vacuum state for the inflaton field *depends* on the amplitude and dynamics of the time-dependent mean field $\hat{\phi}$ [63]. In addition, it is well known that for a quantum field in a (homogeneous and isotropic) Friedmann–Robertson–Walker spacetime, the vacuum state depends on the scale factor. In both cases, even if the background field $\hat{\phi}$ and scale factor a are constant in the asymptotic past and future (conditions which do not appear to hold for realistic inflationary scenarios), the vacuum states defined in the asymptotic past, $|0, \text{in}\rangle$, and in the asymptotic future, $|0, \text{out}\rangle$, are physically inequivalent.

As is well known, the Schwinger–DeWitt path integral formulation of the generating functional $Z[J]$ for n-point functions yields "in–out" matrix elements such as $\langle 0, \text{out}|\Phi_{\text{H}}(x_1)|\text{in}\rangle$, rather than expectation values in terms of the "in" vacuum [30–33]. In such cases, the effective action obtained by the Legendre transform of $-i \log Z[J]$ yields equations of motion for the "classical field" $\hat{\phi}$ whose solutions are neither real nor causally related to J. For this reason, the Schwinger–DeWitt "in–out" formulation of quantum field theory is inconvenient for deriving evolution equations for expectation values of quantum field operators in the presence of a time-dependent background (in the present case, the scale factor and the mean field). Furthermore, when a Cauchy formulation of the coupled dynamics of the spacetime and matter fields is *required*, the Schwinger–DeWitt formalism is unsuitable because it requires a knowledge of the solution for the metric and background field in the asymptotic future before the asymptotic-future boundary conditions on the functional integral (and thereby, the "out" vacuum) can be made well defined.

The Schwinger–Keldysh "closed-time-path" (CTP) formalism provides an elegant and powerful method of deriving evolution equations for expectation values for nonequilibrium quantum fields in a dynamical background [34–41]. One formulates field theory on a spacetime manifold which consists of two copies of the original spacetime manifold which are identified at some spacelike hypersurface Σ_* which is far to the future of any dynamics in which one is interested. This identified manifold inherits an orientation from the original spacetime manifold by reversing the sign of the volume form between the two copies of the spacetime. In this sense, the two copies of the spacetime manifold are different "time branches," designated $+$ and $-$. The direction of time (i.e., the direction of future-directed timelike vectors) on the $-$ time branch is therefore reversed with respect to the $+$ time branch. The functional integral for quantum fields then consists of a sum over c-number field configurations which can be independently specified on the $+$

and $-$ time branches, subject to the constraint that the configurations agree on the Cauchy hypersurface Σ_*. Similarly, the generating functional for n-point functions is a functional of a c-number source J which can be independently specified on the $+$ and $-$ time branches, J_+ and J_-. Anti-time-ordered and time-ordered n-point functions are obtained by n-times differentiating $Z[J_+, J_-]$ with respect to J_- and J_+, respectively, and then setting $J_\pm = 0$. It should be emphasized that the n-point functions obtained by this procedure are true expectation values with respect to the "in" quantum state specified by the boundary conditions on the functional integral. These boundary conditions need only be specified in the asymptotic past on both time branches, thereby permitting a well-posed initial value problem for nonequilibrium quantum fields [39, 40].

Although it has only recently been formulated as a problem involving nonequilibrium dynamics of quantum fields in curved spacetime, the reheating problem in inflationary cosmology was first considered over 16 years ago [42]. Since then, work on the reheating problem, broadly speaking, has followed two distinct approaches, each developing in two stages.

The first approach to the reheating problem assumed a damped, phenomenological equation for the inflaton mean field $\hat{\phi}$,

$$\ddot{\phi} + m^2\phi + (\Gamma + 3H)\dot{\phi} = 0 \tag{1.1}$$

where Γ, given by the imaginary part of the self-energy of the inflaton field ϕ, is the total perturbative decay rate. The symbol H is the Hubble parameter, and dots denote differentiation with respect to cosmological or comoving time. In a first stage of work [42–45], time-dependent perturbation theory was used to compute the rate of particle production Γ into light fields (usually fermions) coupled to the inflaton. Particle production rates were computed in flat space and assuming an eternally sinusoidally oscillating inflaton mean field. This neglects the effect of the time-dependent amplitude of the inflaton mean field on the particle production process. Bose enhancement of particle production rates into the modes of the inflaton fluctuation field φ (and light Bose fields coupled to the inflaton) was not taken into account.

In the second stage of this first approach to the reheating problem [11, 29, 46, 47], Eq. (1.1) was still utilized to model the mean-field dynamics, but Γ was computed beyond first-order in perturbation theory. In the work of Shtanov *et al.* [29] and Kofman *et al.* (KLS) [11], Γ was computed for a real self-interacting scalar inflaton field, taking into account the fact that the mean field appears quadratically in the one-loop mode equations for the inflaton and other fields coupled to it. Approximate expressions for the growth rate of occupation numbers were derived, assuming a quasioscillatory mean field. For bosonic fields coupled to the inflaton, it was found that first-order time-dependent perturbation theory drastically underestimates the particle production rate for modes which are in an instability band (due to parametric resonance). Parametric amplification of quantum fluctuations in bosonic degrees of freedom can result in rapid out-of-equilibrium transfer of energy from the inflaton mean field to the inhomogeneous inflaton modes and light Bose fields coupled to the inflaton. This phenomenon was called *preheating* by KLS. It was noted that the preheating effect can, assuming rapid equilibration, lead to a much larger estimate of the reheating temperature than that given by the fermions-only, perturbative approach (the so-called "elementary" theory of reheating).

In both stages of this first approach, the backreaction of the inflaton variance on the mean-field dynamics and of the variance on the quantum mode functions were not treated self-consistently. Backreaction of the variance on the quantum modes is important to shutting off the preheating process [19]. The effect of spacetime dynamics was either excluded entirely or not included self-consistently using the semiclassical Einstein equation. Due to the potentially large initial inflaton amplitude at the onset of reheating, particularly in the case of chaotic inflation (where the inflaton amplitude at the end of the slow-roll period can be as large as $M_P/3$, [5]), the effect of cosmic expansion on quantum particle production needs to be included. Since the mean field and variance are coupled, the backreaction of particle production on the mean-field dynamics must be accounted for in a self-consistent manner.

The second approach to the postinflationary reheating problem is built upon the body of earlier work on cosmological particle creation [48–52]. Following the application of closed-time-path techniques to nonequilibrium relativistic field theory problems [40, 53], several authors derived perturbative mean-field equations for a scalar inflaton with various self-couplings and couplings to fermion fields [54–57]. The closed-time-path method yields a real and causal mean-field equation with backreaction from quantum particle creation taken into account. For the case of Bose particle production, perturbation theory in the coupling constant is known to break down for sufficiently large occupation numbers. It is therefore necessary to employ nonperturbative techniques in order to study reheating in most inflationary models.

The second stage of work in this second approach to the reheating problem used nonequilibrium methods to derive self-consistent mean-field equations for an inflaton coupled to lighter quantum fields [21–23, 58–62]. In the first of these studies [21–23, 58], the coupled one-loop mean-field and mode-function equations were solved numerically in Minkowski space, implicitly carrying out an *ad hoc* nonperturbative resummation in $\hbar$. In the one-loop equations, the variances for the inflaton $\langle\varphi^2\rangle$ and light Bose fields $\langle\chi^2\rangle$ do not backreact on the mode functions directly. However, mean-field equations were derived for an $O(N)$-invariant linear σ model (with a $\lambda\Phi^4$ self-interaction) at leading order in the large-N approximation by Boyanovsky *et al.* [20]. In this approximation, which includes the two-loop "double bubble" diagram, but not the two-loop "setting sun" diagram [63], the variance does backreact on the quantum mode functions. At leading order in the $1/N$ expansion, the unbroken symmetry dynamical equations for the quartic $O(N)$ model are formally similar to the dynamical equations for a single $\lambda\Phi^4$ field theory in the time-dependent Hartree–Fock approximation [64]. The nonequilibrium dynamics of the quartically self-interacting $O(N)$ field theory in Minkowski space has been numerically studied at leading order in the $1/N$ expansion in both the unbroken symmetry [19, 20, 65] and symmetry-broken [19, 20, 66] cases. Some analytic work has been done on the self-consistent Hartree–Fock mean-field equations for a quartic scalar field in Minkowski space [67]. In addition, the Hartree–Fock equations for a $\lambda\Phi^4$ field in the slow-roll regime have been studied numerically in Minkowski space [68] and in FRW spacetime [69]. However, the effect of spacetime dynamics on reheating in the $O(N)$ field theory has not (to our knowledge) been studied using the coupled, self-consistent semiclassical Einstein equation and matter-field dynamical equations, though some simple analytic work has been done on curvature effects in reheating [46, 59]. The semiclassical equations for one-loop reheating in FRW spacetime were derived in ref. 60. The $\phi^2\chi^2$ theory has been studied in FRW spacetime by in refs. 70, 61, and 62.

In this paper we summarize a recent study [63, 27] which focuses on the effect of cosmic expansion on the nonequilibrium dynamics of the inflaton field during postinflationary reheating, and in particular on the "preheating" mechanism of parametric resonance-induced energy transfer to the inhomogeneous inflaton modes. While we assumed a chaotic inflation picture [71] which avoids many of the technical difficulties (such as infrared divergences) which can arise in the study of phase transitions in curved spacetime [72–74], the formalism utilized in our work can also be applied "new" inflation models with a symmetry-breaking inflaton potential. A more thorough discussion of this work, along with an introduction to nonequilibrium quantum field theory in curved spacetime, can be found in ref. 75.

We follow the sign conventions of Birrell and Davies [33] for the metric tensor $g_{\mu\nu}$ and the Riemann tensor $R_{\alpha\beta\gamma\delta}$. In this convention, the metric signature is $(+, -, -, -)$. We use the beginning Latin indices a, b, c, d to denote closed-time-path "time path" indices (with an index set $\{+, -\}$), and the Latin indices i, j, k, l for the $O(N)$ space. Greek letters are used to denote spacetime indices. We work in units where $\hbar = c = k_B = 1$.

2. EFFECT OF COSMIC EXPANSION ON PREHEATING

For a study of the effect of cosmic expansion on the parametric resonance-induced transfer of energy from the inflaton mean field to inhomogeneous quantum modes, it is sufficient to consider only particle production to the inflaton's inhomogeneous quantum modes. Because backreaction of the inflaton variance on the mean field and inhomogeneous modes is important during preheating, a nonperturbative truncation of the Schwinger–Dyson equations is required. In a model with a global O(N) symmetry, the large-N expansion at leading order provides such a truncation [65]. At this order, the only inflaton correlation functions which are dynamical are the mean field and the variance. As we are interested in the nonequilibrium dynamics of this model, we specify Cauchy data for the truncated theory, consistent with the end state of slow roll, at some initial spatial hypersurface Σ_0.

We start with a classical model consisting of a scalar inflaton field Φ^i with quartic self-coupling and a global O(N) symmetry in a classical background spacetime. The classical action for the theory is

$$S[\phi^i, g^{\mu\nu}] = S^{\mathrm{G}}[g^{\mu\nu}] + S^{\mathrm{F}}[\phi^i, g^{\mu\nu}] \tag{2.1}$$

where the classical gravity action S^{G} is

$$S^{\mathrm{G}}[g^{\mu\nu}] = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,[R - 2\Lambda + cR^2 + bR^{\alpha\beta}R_{\alpha\beta} + aR^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}] \tag{2.2}$$

and the matter action S^{M} is given by

$$S^{\mathrm{F}}[\phi^i, g_{\mu\nu}] = -\frac{1}{2}\int_M d^4x\,\sqrt{-g}\left[\bar{\vec{\phi}}\cdot(\Box + m^2 + \xi R)\bar{\vec{\phi}} + \frac{\lambda}{4N}(\bar{\vec{\phi}}\cdot\bar{\vec{\phi}})^2\right] \tag{2.3}$$

Here, g denotes the determinant of the metric tensor $g_{\mu\nu}$, $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor, $R_{\alpha\beta}$ is the Ricci tensor, R is the scalar curvature, G is Newton's constant, and $\Box$ is the Laplace–Beltrami operator. The constants[2] a, b, and c have dimensions of length squared, and the cosmological constant Λ has dimensions inverse length-squared; all four parameters will be set to zero after renormalization of the right-hand side of the semiclassical Einstein equation. The dimensionless bare parameters ξ and λ are the conformal coupling to gravity and the coupling constant of the O(N) theory, respectively. The parameter m is the bare mass of the theory, where m^2 is positive, so that the O(N) symmetry is unbroken. The inflaton field is quantized on the classical background spacetime, with an initial quantum state $|\Omega\rangle$ (specified

[2]The higher curvature terms must be included in the bare gravitational action in order to renormalize the divergences which arise in the expectation value of the quantum energy-momentum tensor $\langle T_{\mu\nu}\rangle$.

at Σ_0) which has a nonzero expectation value for the Heisenberg field operator $\Phi^i_{\rm H}$,

$$\hat{\phi}^i \equiv \langle \Omega | \Phi^i_{\rm H} | \Omega \rangle \tag{2.4}$$

for some value of i. The quantum state $|\Omega\rangle$ is defined in terms of the adiabatic vacuum state (matched in the asymptotic past) for the fluctuations $\varphi^i_{\rm H}$ about the mean field

$$\varphi^i_{\rm H} \equiv \Phi^i_{\rm H} - \hat{\phi}^i \tag{2.5}$$

In our study, the dynamics of the background spacetime is determined self-consistently by the semiclassical Einstein equation,

$$\frac{2}{\sqrt{-g}} \frac{\delta S^{\rm G}}{\delta g^{\mu\nu}} = -8\pi G \langle T_{\mu\nu} \rangle_{\rm ren} \tag{2.6}$$

where the "ren" subscript denotes that divergences on the right-hand side are understood to be suitably regularized within the context of a covariant renormalization procedure. The symbol $\langle T_{\mu\nu} \rangle_{\rm ren}$ represents the expectation value of the classical energy-momentum tensor, with the Heisenberg field operator $\Phi^i_{\rm H}$ replacing the classical field ϕ^i. Henceforth, all expectation values are with respect to the quantum state $|\Omega\rangle$. The same truncation (namely, leading-order in the large-N expansion) is applied to the expectation value of the energy-momentum tensor as to the Schwinger–Dyson equations for correlation functions of the quantum field.[3]

For consistency with the spatial homogeneity and near-spatial flatness expected at the end of the slow-roll period during inflation (as well as for simplicity), we assume that the background spacetime is spatially flat Friedmann–Robertson–Walker (FRW) spacetime. We write the line element in the form

$$ds^2 = a(\eta)^2 \left[d\eta^2 - \sum_{i=1}^{3} (dx^i)^2 \right] \tag{2.7}$$

where a is the scale factor and x^i ($i \in \{1, 2, 3\}$) are the physical position coordinates on the spatial hypersurfaces of constant conformal time η (related to the cosmic time t by $\eta = \int dt/a$). We shall be specifying initial data on a spacelike hypersurface of constant $\eta = \eta_0$.

The spatial homogeneity and isotropy of FRW spacetime permit only two algebraically independent components of the energy-momentum tensor, which in the FRW coordinates of Eq. (2.7) are given by $\langle T_{00} \rangle$ and $\langle T_{ii} \rangle$; all

[3] The semiclassical approximation for gravity is consistent with a truncation of the quantum theory of matter plus gravity perturbations at leading order in the large-N expansion [76, 77].

other components are zero. These must be functions of η only, due to spatial homogeneity. For the purpose of numerically solving the semiclassical Einstein equation, it is convenient to work with the trace

$$\mathcal{T} = g^{\mu\nu}\langle T_{\mu\nu}\rangle = a^{-2}\eta^{\mu\nu}\langle T_{\mu\nu}\rangle \tag{2.8}$$

instead of $\langle T_{ii}\rangle$. The trace $\mathcal{T}$ enters into the dynamical equation for $a(\eta)$, and $\langle T_{00}\rangle$ enters into the constraint equation.

Another consequence of the spatial symmetries of the FRW spacetime is that the mean field $\hat{\phi}$ must be spatially homogeneous at $\eta = \eta_0$, i.e.,

$$\hat{\phi}^i(\eta_0, \bar{x}) = \hat{\phi}^i(\eta_0) \tag{2.9}$$

Because the Lagrangian is spatially translation and rotation invariant, this spatial homogeneity is preserved in the full quantum evolution, so that $\hat{\phi}^i$ is a function of η only, for all time. Similarly, the various equal-time CTP two-point functions [63] $G^{ij}_{ab}(x, x')$ (for $a, b \in \{+, -\}$) are also functions of time and $|\bar{x} - \bar{x}'|$ only,

$$G^{ij}_{ab}(\eta, \bar{x}; \eta, \bar{x}') = G^{ij}_{ab}(\eta, |\bar{x} - \bar{x}'|) \tag{2.10}$$

In the coincidence limit $\bar{x} = \bar{x}'$, the four CTP equal-time two-point functions are all the same, and equal to the *variance* of the inflaton field, which for unbroken symmetry takes the form

$$\langle\Omega|\varphi^i_{\mathrm{H}}(\eta, \bar{x})\varphi^i_{\mathrm{H}}(\eta, \bar{x})|\Omega\rangle = G^{ij}_{ab}(\eta, \bar{x}; \eta, \bar{x}) \tag{2.11}$$

We choose initial conditions for the metric which are consistent with a spacetime which is asymptotically de Sitter at $\eta \to -\infty$, so that for $\eta < \eta_0$,

$$a(\eta) \simeq \frac{1}{1 + H(\eta)(\eta - \eta_0)} \tag{2.12}$$

where the parameter $H(\eta)$ is a slowly varying function of time given by

$$H(\eta) \equiv \sqrt{\frac{8\pi G T^{\mathrm{C}}_{00}(\hat{\phi}^i)}{3a^2(\eta)}} \tag{2.13}$$

and $T^{\mathrm{C}}_{00}(\hat{\phi}^i)$ is the 0–0 component of the classical energy-momentum tensor for the inflaton mean field. With $\hat{\phi}^i(\eta)$ governed (approximately) by the classical slow-roll equation of motion

$$(\hat{\phi}^i)'' + \frac{2a'}{a}(\hat{\phi}^i)' + a^2\left(m^2 + \frac{\lambda}{N}\hat{\phi}^j\hat{\phi}^k\,\delta_{kj}\right)\hat{\phi}^i = 0 \tag{2.14}$$

the metric for $\eta < \eta_0$ describes the slow-roll dynamics of an inflaton field which has been "eternally inflating" (i.e., the spacetime is asymptotically de

Sitter as $\eta \to -\infty$). Primes in Eq. (2.14) denote differentiation with respect to conformal time η.

In taking the large-N limit, for the case of unbroken symmetry, the initial expectation value $\hat{\phi}^i(\eta_0)$ can be in any direction in the O(N) space; we choose it to be $i = 0$. Because of the O(N) invariance of the theory, the physics does not depend on the direction we choose. We can then define

$$\hat{\phi}(\eta) \equiv \hat{\phi}^0(\eta) \tag{2.15}$$

In taking the large-N limit, we rescale the Planck mass M_P by $\sqrt{N}$, because $\langle T_{\mu\nu}\rangle_{ren} \sim O(N)$. In the large-$N$ expansion truncated at leading order, the equation of motion for $\hat{\phi}$ in spatially flat FRW spacetime becomes

$$\hat{\phi}'' + \frac{2a'}{a}\hat{\phi}' + a^2 M^2(\eta)\hat{\phi} = 0 \tag{2.16}$$

where $M^2(\eta)$ is defined by

$$M^2(\eta) \equiv m^2 + \frac{\lambda}{2}\hat{\phi}^2(\eta) + \frac{\lambda}{2}\langle\varphi_H^2(\eta)\rangle \tag{2.17}$$

and the variance $\langle\varphi_H^2(\eta)\rangle$ is spatially translation and rotation invariant, and given by

$$\langle\varphi_H^i(\eta, \bar{x})\varphi_H^j(\eta, \bar{x})\rangle = \delta^{ij}\langle\varphi_H(\eta)^2\rangle \tag{2.18}$$

The bare variance can be expressed in terms of an integral over spatial momenta of the modulus squared of the conformal mode functions $\tilde{u}_k$ in which the Heisenberg field operators are expanded [27],

$$\langle\varphi_H^2(\eta)\rangle = \frac{1}{a^2}\int\frac{d^3k}{(2\pi)^3}|\tilde{u}_k(\eta)|^2 \tag{2.19}$$

The complex mode functions $\tilde{u}_k(\eta)$ obey a harmonic oscillator-type equation with a time-dependent frequency $\Omega_k(\eta)$,

$$\tilde{u}_k''(\eta) + \Omega_k^2(\eta)\tilde{u}_k(\eta) = 0 \tag{2.20}$$

where $\Omega_k(\eta)$ is given by

$$\Omega_k(\eta)^2 \equiv k^2 + a^2(\eta)\left[M^2(\eta) + \left(\xi - \frac{1}{6}\right)R(\eta)\right] \tag{2.21}$$

and $R(\eta)$ is the curvature scalar,

$$R(\eta) = \frac{6a''}{a^3} \tag{2.22}$$

The quantum energy-momentum tensor, at leading order in the large-N approximation, has classical and quantum pieces,

$$\langle T_{\mu\nu}(\Phi_{\rm H})\rangle = T^{\rm C}_{\mu\nu}(\hat{\phi}) + T^{\rm Q}_{\mu\nu}[\tilde{u_k}] - \frac{\lambda}{8}\langle\varphi^2_{\rm H}(\eta)\rangle^2 g_{\mu\nu} \tag{2.23}$$

The third term in the above equation is also quantum in origin, but for convenience of notation, $T^{\rm Q}_{\mu\nu}$ is defined so as not to include this term [63]. Expressions for the classical piece $T^{\rm C}_{\mu\nu}(\hat{\phi})$ and $T^{\rm Q}_{\mu\nu}[\tilde{u_k}]$ can be found in Eqs. (3.11) and (3.12) of ref. 27. The energy density $\rho_{\rm Q}$ of the inhomogeneous quantum modes of the inflaton field is given by

$$\rho_{\rm Q} \equiv \frac{1}{a^2} T^{\rm Q}_{00} - \frac{\lambda}{8}\langle\varphi^2_{\rm H}(\eta)\rangle^2 \tag{2.24}$$

Equation (2.19) for the variance is divergent in three spatial dimensions, and if one imposes a spatial momentum cutoff K, it is proportional to K^2. Similarly, the 0–0 component of the quantum part of the energy-momentum tensor is proportional to K^4, and the trace is proportional to K^2. For quantum fields in an expanding universe, adiabatic regularization can be used to obtain a covariantly conserved, finite energy-momentum tensor [52, 78–82]. Adiabatic regularization is carried out in the context of a renormalization of G, a, b, c, Λ, ξ, λ, and m [27]. In place of the bare variance and the bare energy-momentum tensor which appear in the mode function and semiclassical Einstein equations, after regularization one has

$$\langle\varphi^2_{\rm H}(\eta)\rangle_{\rm ren} \equiv \langle\varphi^2_{\rm H}(\eta)\rangle - \langle\varphi^2_{\rm H}(\eta)\rangle_{\rm ad2} \tag{2.25a}$$

$$(T^{\rm Q}_{\mu\nu})_{\rm ren} \equiv T^{\rm Q}_{\mu\nu} - (T^{\rm Q}_{\mu\nu})_{\rm ad4} \tag{2.25b}$$

The quantities without subscripts are the bare expressions for the variance and quantum energy-momentum tensor, and the quantities with "ad2" and "ad4" subscripts denote second- and fourth-adiabatic order expansions, respectively, of the variance and quantum energy-momentum tensor, computed using a WKB-type *Ansatz* for the mode functions [27]. In order to compute numerically the mode integrals in the renormalized variance and energy-momentum tensor, an ultraviolet spatial momentum cutoff must be imposed [65]. In order that the relation between the bare and renormalized mass is not time dependent, it is necessary that the ultraviolet cutoff K be imposed in terms of a maximum *physical* momentum, rather than *comoving* momentum [83]. For sufficiently large K, the adiabatically regularized variance and energy-momentum tensor are cutoff independent [27].

Initial data for the leading-order large-N equations and the semiclassical Einstein equation consist of $\hat{\phi}(\eta_0)$, $\hat{\phi}'(\eta_0)$, $a(\eta_0)$, and initial values for $\tilde{u_k}(\eta_0)$ and $\tilde{u}'_k(\eta_0)$, for all spatial momenta k. The value $a'(\eta_0)$ is fixed by the 0–0 component of the semiclassical Einstein equation, which is a constraint. We choose $a(\eta_0) = 1$, $\hat{\phi}'(\eta_0) = 0$, and $\lambda\hat{\phi}(\eta_0)^2/4 = m^2$. The initial data for the modes $\tilde{u}_k$ are chosen to coincide with the zeroth-order adiabatic vacuum (matched at $\eta = -\infty$), evolved forward in time to the initial-data hypersurface $\eta = \eta_0$. The adiabatic vacuum is the best approximation to a no-particle state as would be measured by a comoving detector coupled to the quantum field [78, 33, 84]. This corresponds to the following choice for initial data for the modes:

$$\tilde{u}_k(\eta_0) = \left(\frac{-\pi}{4H_0}\right)^{1/2} H_\nu^{(2)}(-kH_0^{-1}) \tag{2.26a}$$

$$\tilde{u}'_k(\eta_0) = \frac{d}{d\eta}\left[\left(\frac{\pi\eta}{4}\right)^{1/2} H_\nu^{(2)}(k\eta)\right]_{|\eta=-H_0^{-1}} \tag{2.26b}$$

where $H_0 \equiv H(\eta_0)$, and $H_\nu^{(2)}$ is the Hankel function of second kind.

The value $\lambda = 10^{-14}$ was chosen to be representative of typical chaotic inflation models with a quartic potential, and to avoid inconsistency with the observed fluctuations in the cosmic microwave background [29]. The minimal coupling case, $\xi = 0$, was chosen for this study (note, however, ref. 85, where a nonminimal coupling of the inflaton is considered). Energy units where $m = 1$ were chosen, for simplicity in carrying out the numerical solution to the dynamical equations. In these units, $\hat{\phi}(\eta_0) = 2.0 \times 10^7$. Values of the momentum cutoff were chosen between $K = 50$ and $K = 70$ in order to verify that the results of the numerical calculation are cutoff independent. The only remaining parameter is the value of the Planck mass in the chosen energy units, which we varied by choosing different values for the ratio $M_P/\hat{\phi}(\eta_0)$. Values chosen for this ratio were 5×10^6, 5×10^4, 3×10^3, and 300. Roughly one-third this ratio is the initial value of the inverse Hubble constant; thus, the smaller the ratio $M_P/\hat{\phi}(\eta_0)$, the more rapid is the initial rate of cosmic expansion. With the value chosen for λ and the adiabatic vacuum initial conditions, the initial value for the contribution of the variance to the effective mass, $\lambda\langle\varphi^2\rangle/2$, is very small, $\sim 10^{-16}$. As a result of the oscillations of the inflaton mean field $\hat{\phi}$, parametric resonance will cause low-k modes of the inflaton field (which are said to be in a "resonance band") to grow exponentially. Given the values chosen for the parameters $\hat{\phi}(\eta_0)$ and λ, for very slow cosmic expansion, the analytic estimate of ref. 19 gives the time scale for the quantity $\lambda\langle\varphi^2\rangle/2$ to grow to be of order unity; in our model, this *preheatinq time scale*, which we denote by τ_1, is about 133.

The results of our numerical solution of the coupled dynamical equations for the metric and inflaton field show that cosmic expansion can dramatically effect the preheating process when the initial Hubble time scale is less than or equal to the preheating time scale. We find that when the Hubble time scale $H(\eta_0)^{-1} > \tau_1$, preheating is an efficient mechanism of energy transfer

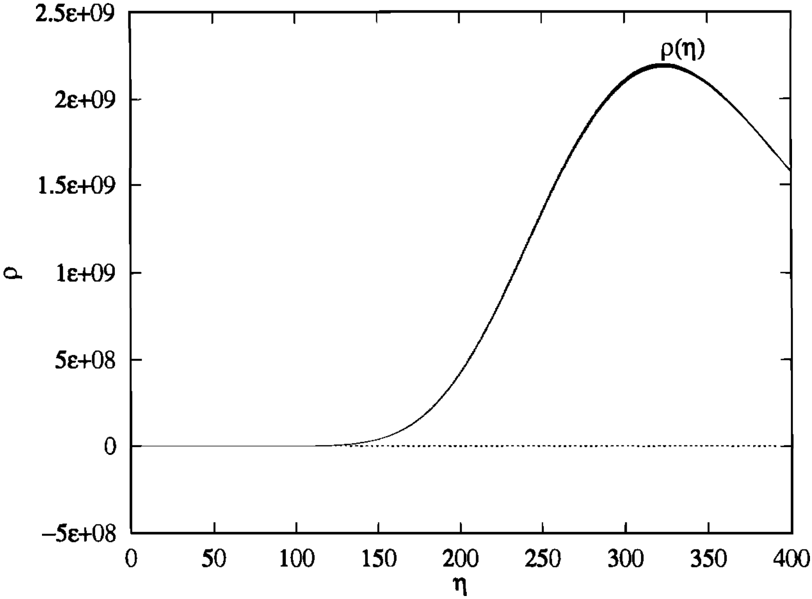

Fig. 1.

from the inflaton mean field to the inhomogeneous modes. For cases of *very* slow expansion, $H(\eta_0)^{-1} \gg \tau_1$, our results agree with simulations of the leading-order, large-N dynamics of a global $O(N)$ scalar field theory in Minkowski space [19], with unbroken symmetry. For the case of rapid expansion, $H(\eta_0)^{-1} \lesssim \tau_1$, we find that preheating is *not* efficient, i.e., the term $\lambda\langle\varphi^2\rangle/2$ is never of order unity. This can be seen from a plot of the quantum energy density ρ_Q versus conformal time, for the rapid-expansion case of $M_P/\hat{\phi}(\eta_0) = 300$, where $H^{-1}(\eta_0) \simeq 104$. This plot is shown in Fig. 1.

In Fig. 1 it is clear that the energy density in inhomogeneous quantum modes of the inflaton field never grows to be of the order of the energy density of the classical field, which is of order 4×10^{14}. This provides clear evidence that in this model, preheating is not efficient when $M_P/\hat{\phi}(\eta_0) \lesssim 300$.

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